Mathematical Modelling in Science and Engineering using Fractional Calculus

A Synopsis for the proposed Ph.D. thesis in Mathematics.

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1. STATE OF ART:

1.1. Introduction

Mathematical modelling is a branch of mathematics which is being applied in many fields of research like Science & Engineering, Commerce, Finance, Economics, Physics, Biology, Medicine and many others interdisciplinary branches. It gives a framework to understand current complex problems, predict about their future, and provides flexibility to simulate different situations which have not occurred yet (Kapur, 2023). This advantage of mathematical modelling makes it powerful and makes research more cost effective. Traditional models, often expressed through classical calculus, have been foundational for centuries. However, as systems become more intricate—spanning diverse and interconnected fields, there is a growing recognition of the limitations of integer-order models in capturing memory effects, hereditary properties, and anomalous diffusion. But in the last 3-4 decades, fractional calculus has emerged as a solution to this and this makes mathematical models more realistic and relevant. Like other fields, Mathematical Modelling has also played vital role in development of Epidemiology- the branch of medicine which studies transmission of diseases in different population.

Recently, we all have seen the Covid 19 pandemic and its tragedies- deaths, lockdown, economic disasters etc. Like Covid-19, there are other infections. Monkeypox (mpox) is such an example. Monkeypox is emerging as a new threat to the world. WHO (World Health Organization) declares mpox virus a public health emergency of international concern (https://news.un.org/en/story/2024/08/1153176). As on June 23, 2022, the World Health Organization designated monkeypox as an "emerging risk of moderate public health concern." More than 65,000 cases of monkeypox virus infection have been reported globally in 106 countries and five geographic zones since September 2022. Understanding of transmission dynamics of such infections plays important role in controlling it and minimizing its tragedies.

Fractional calculus, a generalization of classical calculus, introduces derivatives and integrals of non-integer (fractional) orders. This powerful mathematical tool extends the modelling capabilities, allowing more accurate descriptions of systems with complex dynamics, long-term dependencies, and non-local behaviors. Fractional models offer greater flexibility in describing real-world phenomena, making them particularly useful in fields such as viscoelasticity, electromagnetics, fluid dynamics, and control theory.

In this proposed work, we aim to explore how fractional calculus enhances mathematical modelling of monkeypox virus transmission dynamics. We delve into the fundamental principles of fractional derivatives and apply it to monkeypox virus transmission dynamics to understand memory effects in its transmission. This will make monkeypox virus transmission dynamics more realistic, which in turn highlights the increasing relevance of fractional-order models in tackling contemporary real-world challenges of monkeypox virus

and advancing use of scientific knowledge in different field of science to make society sustainable and ready for future threats.

1.2. Fractional Order Epidemic models- Monkeypox virus transmission Dynamics

Epidemic model- Epidemic mathematical models are essential tools for understanding the spread of infectious diseases within populations, predicting future outbreaks, and informing public health interventions. These models represent the dynamics of disease transmission using mathematical equations and can help policymakers decide how to allocate resources, implement vaccination programs, or impose social distancing measures, (Barnes & Fulford, 2014). Several types of epidemic models exist, each capturing different aspects of disease transmission. Recently we have witnessed the Covid-19 pandemic and we know the losses due to such epidemic. The monkey pox is also similar threat to the humanity.

1.2.1.Monkeypox

WHO (World Heath Organization) declares mpox virus a public health emergency of international concern. The monkeypox virus is a member of the orthopoxvirus genus in the Pxviridae family and causes monkeypox. The incubation time for monkeypox is 6-16 days which may extend 5-21 days and symptoms are fever, headache, muscular ache, backache, swollen lymph nodes. The symptoms are visible in two stages – for first stage, incubation time is 5 days where main sign like fever comes and for second stage, incubation time is 11 days where symptoms like lymph node swelling, back pain, extreme head ache come. The death rate is upto10% and the most vulnerable to monkeypox are children under age of 10. Till date no clear treatment is available for the monkeypox infection. But some vaccine like brincindefovir, tecovirimat etc. are used in this infection (Peter et al., 2022).

The first known monkeypox virus was reported in the Democratic Republic of the Congo in 1970. On June 23, 2022, the World Health Organization designated monkeypox as an "emerging risk of moderate public health concern." More than 65,000 cases of monkeypox virus infection have been reported globally in 106 countries and five geographic zones since September 2022, resulting in 26 deaths, so Monkey pox as alarming threat. Primarily, wild animals like African rats and monkeys transmit the virus to people, however human to human transmissions are also frequent in most of the reported cases. Animal to human transmission occurs as a result of bites or scratches, direct contact or consuming food contaminated by rodents. Our readiness will save many lives.

1.2.2.Mathematical Model of Monkeypox Dynamics

There have been few studies on monkeypox transmission and there are some mathematical models for transmission of MPXV using classical calculus. Peter et al. (2022) have proposed a deterministic model and analysed the transmission or outbreak of monkeypox virus using classical calculus. In this work they have considered both human population and rodent for transmission. Also, they have considered effect of 5 different human population

classes like susceptible, exposed, infected, isolated and recovered. Bhunu and Mushayabasa (2011) have also given a deterministic model where only susceptible, infectives and recovered human population were considered.

Fractional calculus is increasingly used in epidemic modelling due to its ability to incorporate memory effects, long-range spatial interactions, and heterogeneity in disease transmission dynamics (Miller & Ross, 1993). Traditional models based on integer-order derivatives, such as the SIR (Susceptible-Infectious-Recovered) and SEIR (Susceptible-Exposed-Infectious-Recovered) models, often assume that changes in compartments (e.g., infection or recovery) depend only on the current state. However, many real-world epidemiological processes like spread of covid -19 virus, monkeypox etc exhibit **history-dependent dynamics** (e.g., immune response, incubation periods), which can be captured more accurately with fractional derivatives. This gives a Dynamical system which involve nonlinear differentia equations due to factors -incubation period of the disease and population interactions and vaccination effects. Stability, Bifurcations and Chaos analysis are important components to be analysed for nonlinear dynamical systems (Perko,2021; Strogatz,1994).

There are few works on modelling monkeypox virus transmission using Fractional order derivatives. Peter et al. (2022) in their work predicted that the memory index or fractional order can also be used as a control parameter but not explained how can be it done. Okyere and Ackoraa-Prah (2023) in their work have modelled the monkeypox transmission in Ghana using fractional calculus. They highlighted that fractional operator had an influence on the various compartments of the model. The dynamics of the disease in the community were shown to be influenced by fractional-order derivatives, and infections were eradicated within the first five (5) days when $\pi = 0.2$. In this work, authors have considered only human to human transmission, effect of rodent population is not considered.

Research Gap- Modelling Monkeypox virus transmission using fractional derivative where effect of both human population and rodent population is not done yet and relation of the dynamics with orders of fractional differential equation.

2. PROBLEM TO BE INVESTIGATED

In this proposed work , we aim to formulate mathematical model using fractional calculus for the monkeypox virus transmission considering both human and rodent population. We will also aim to analyze effect of the order of the derivative on money pox virus transmission. We will include stability analysis, bifurcation study, limit cycle, equilibrium points, linearization of the system of equations for monkeypox transmission dynamics and investigation of the appropriate solution techniques of the model.

3. OBJECTIVE OF THE INVESTIGATION AND MOTIVATION

We faced Covid-19 like pandemic recently. As per WHO data, 7.1 million people died due to covid-19, which is a huge number. Initially Covid-19 virus spreading behaviors could not be predicted, so governments have to impose the lock-downs. Later some Models were developed but these models could predict the behavior at some extent. Use of Fractional calculus played crucial role in this direction. This gave a motivation to investigate other models using fractional calculus to be more realistic and appropriate insights. This will prepare our society against these type of disasters as WHO declares monkeypox as an alarming threat.

4. RESEARCH METHODOLOGY

4.1 Methods

In this proposed work, we will use fractional calculus to monkeypox virus transmission dynamics. This will give fractional differential equations. We will use appropriate numerical and analytical methods to solve and analyze the obtained fractional differential equations.

4.1.1. Fractional Calculus-

In classical calculus we use integer order derivatives and integration. The generalization of integral order derivatives and integrations to arbitrary order is coined as fractional calculus. The beginning of the fractional calculus (Miller and Ross ,1993) is the answer to the G. W. Leibniz's letter to L' Hospital in 1695 raising the following question: "*Can the meaning of derivatives with integer order be generalized to derivatives with non-integer orders?*". In

addition, L' Hospital replied to Leibniz by another question: "*What if the order will be 1/2?*", then Leibniz replied through another letter dated September 30, 1695, the exact birth of the fractional calculus! "*It will lead to a paradox, from which one day useful consequences will be drawn*". After this question, the fractional derivative has been an ongoing topic for more than 300 years (Podlubny,1999).

Fractional Calculus has numerous applications in mathematical modelling-specially in Epidemic models, modelling virus like monkeypox, diffusion etc.

4.2. Some Terms Used in mathematical modelling using Fractional Calculus.

FDE- Fractional Differential Equations. The differential equation involving fractional or non-integer order derivative is called FDE. For Examples -

4.2.1. Grunwald-Letnikov Fractional Derivative

The Grunwald-Letnikov fractional derivative of order $p > 0$ of a function $f(x)$ is defined as

$$ {}_aD_t^p f(t) = \lim_{\substack{h\to 0 \\ nh=t-a}} h^{-p} \sum_{r=0}^{n} (-1)^r \binom{p}{r} f(t-rh) $$

where

$$ \binom{p}{r} = \frac{p(p-1)(p-2)(p-3)(p-4)\dots(p-r+1)}{r!} = \frac{\Gamma(p+1)}{r!\,\Gamma(p-r+1)} $$

4.2.2. Riemann-Liouville fractional derivative

The Riemann-Liouville fractional derivative is defined as

$$ {}_aD_t^p f(t) = \left(\frac{d}{dt}\right)^{m+1} \int_a^t (t-z)^{m-p} f(z)dz, \qquad (m \le p < m+1). $$

4.2.3. Caputo's Fractional Derivative

$$ {}_a^C D_t^\alpha f(t) = \frac{1}{\Gamma(\alpha-n)} \int_\alpha^t \frac{f^{(n)}(z)dz}{(t-\alpha)^{\alpha+1-n}} \qquad (n-1 < \alpha < n), \text{ (Podlubny, 1999).} $$

4.3 Software to be used.

Primarily the MATLAB software will be used to solve the fractional order differential equation, to plot the graph, to analyse other relevant aspect of the models.

5.CONTRIBUTION TO THE EXISTING KNOWLEDGE.

We have classical mathematical models i.e. mathematical models involving integer order differential equation which does not consider memory factors and global property. Use of Fractional Calculus considers these and gives more realistic prediction or results in the mathematical model analysis. So present work will contribute in making monkeypox virus transmission modelling more realistic.

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